ACADÉMIE DE LA RÉPUBLIQUE SOCIALISTE DE ROUMANIE

REVUE ROUMAINE DE MATHÉMATIQUES PURES ET APPLIQUÉES

TOME XXXIII, N° 9 1988

TIRAGE À PART

EDITURA ACADEMIEI REPUBLICII SOCIALISTE ROMÂNIA

Dedicated to Professor ROMULUS CRISTESCU on the occasion of his 60th birthday anniversary

ALFSEN-EFFROS TYPE ORDER RELATIONS DEFINED BY VECTOR NORMS

CONSTANTIN P. NICULESCU and DAN TUDOR VUZA

INTRODUCTION

The purpose of this paper is to extend some concepts from the M-structure theory of Banach spaces to the setting of Banach spaces endowed with vector norms. The main feature of this approach is that it brings together facts from apparently distinct theories, such as M — structure theory on one side, and Banach lattice theory on the other.

Our paper is divided into 6 sections.

§§ 1—3 have an introductory character. The problem area in this paper can be viewed as a part of the general theory of Alfsen-Effors type order relations developed by the first author; for this reason, § 2 is devoted to a brief survey of some basic facts in that theory. In § 3 we present those facts concerning vector norms and their duality that will be needed in the following sections. Since we cannot give a satisfactory reference for the duality of vector norms, we have included, for the reader's convenience, all details there.

The main concepts in the paper are introduced in § 4. Given a Banach space E and an isometric vector norm $\varphi : E \to X$, where X is a Banach lattice, we may consider the following two order relations of Alfsen-Effros type:

$$x \ll_{L,\varphi} y \text{ if and only if } \varphi(y) = \varphi(x) + \varphi(y - x),$$

$$x \ll_{M,\varphi} y \text{ if and only if } \varphi(z + x) \leqslant \varphi(z) \vee \varphi(z + y) \text{ for every } z \in E.$$

For $\varphi = \|\ \|$, the norm of E, one finds again the relations $\ll_L$ and $\ll_M$ introduced by Alfsen and Effros [1]. For $\varphi = |\ |$, the modulus of a Banach lattice E, one finds that both $\ll_{L,\varphi}$ and $\ll_{M,\varphi}$ coincide with $\ll_o$, a relation of Alfsen-Effros type introduced by the first author in [10].

Various concepts associated with the above defined relations such as centralizers, projections, ideals and summands, are discussed throughout the section 4; in particular, the duality between the centralizers of $\ll_{L,\varphi}$ (respectively $\ll_{M,\varphi}$) and $\ll_{M,\varphi'}$ (respectively $\ll_{L,\varphi'}$) is established; here φ' denotes the dual vector norm of φ.

In §§ 5—6 we realize the announced unification between some results from M-structure theory and Banach lattice theory. Thus, a result of Cunningham, Effros and Roy [7] asserts that every $\ll_M$-summand in a dual Banach space is weak' — closed. A result of Luxemburg and Zaanen [14], [15] asserts that every band in the dual of a Banach lattice with order continuous norm is weak′ — closed. Remarking that $\leqslant_M$-summands and projection bands are particular instances of our notion of an $\leqslant_{M,\varphi}$ summand, it is the purpose of § 5 to give a general theorem which includes both of the above stated results as particular cases.

In the same manner, the purpose of § 6 is to give a unified version of another couple of results: namely, Behrends' result asserting that every Banach space which is an $\leqslant_L$-summand in its second dual is weakly sequentially complete, and Lozanovskii's result asserting that every Banach lattice which is a band in its second dual is weakly sequentially complete. See [3] and respectively [14] for details.

The first named author is much indebted to Professor E. Behrends for providing him with a copy of the monograph [2].

1. PRELIMINARIES

We begin by listing some notations to be used in connection with a Banach space E:

1_E, the identity map on E.

B_E, the closed unit ball in E.

E', the dual Banach space of E.

$\mathcal{J}_E$, the canonical inclusion of E into E''.

The term "w' — topology" will be employed to design any of the weak′ — topologies $\sigma(E', E)$, $\sigma(E'', E')$ and $\sigma(E''', E'')$; the context will be clear enough to understand which is the topology the above term refers to. Convergence with respect to the w' — topology will be denoted by $\xrightarrow{w'}$.

As usual, U' will be the transpose of a bounded linear operator U between two Banach spaces.

Given a vector space E and a seminorm p on E, we denote by (E, p) the Banach space associated to p. This is by definition the completion of the normed vector space $E/p^{-1}(\{0\})$. The canonical map $T : E \to (E, p)$ is by definition the composition of the maps $E \to E/p^{-1}(\{0\}) \to (E, p)$.

We note that all the results in our paper are true for real Banach spaces as well as for complex Banach spaces; however, for the sake of simplicity we consider only real Banach spaces and we only indicate, at the appropriate places, the modifications needed by some definitions in order to cover the complex case.

Given a vector lattice X, we denote by X_x the principal order ideal generated by $x \in X_+$, i.e., the set of those $y \in X$ such that $|y| \leqslant ax$ for some $a \in \mathbb{R}_+$ (depending on y).

The element $e \in X_+$ is called a *strong order unit* provided that $X_e = X$. Whenever X is Archimedean and e is a strong order unit, the norm $\| \ \|_e$ associated to e is defined by

$$\|x\|_e = \inf \{a \mid a \in \mathbb{R}_+, |x| \leqslant ae\}.$$

In particular, every $x \in X_+$ is a strong order unit in the vector lattice X_x; consequently, whenever X is Archimedean we may consider the norm $\| \ \|_x$ on X_x.

e shall make use of the well known fact that every principal order ideal X_x in a Banach lattice X is order isomorphic and isometric (for the norm $\| \ \|_x$) to the Banach lattice $C(K)$ of all continuous real-valued functions on a suitable compact space K.

A *lattice homomorphism* is a linear map U between two vector lattices X, Y such that $U(x_1 \wedge x_2) = U(x_1) \wedge U(x_2)$ for every x, $x_2 \in X$.

Finally, recall that a Banach lattice is said to have *order continuous norm* provided that $\| x_\delta \| \to 0$ whenever $(x_\delta)_\delta \subset X$ is a net such tha t $x_\delta \downarrow 0$

We refer the reader to the monographs [6] and [15] for the elements of vector lattice theory used throughout the paper.

2. ALFSEN-EFFROS TYPE ORDER RELATIONS AND THE CENTRALIZERS ASSOCIATED WITH THEM

2.1 *Definition.* Let E be a Banach space. An order relation $\preccurlyeq$ on E is said to be of *Alfsen-Effros type* provided that the following conditions are satisfied :

i) $u \preccurlyeq v$ implies $v - u \preccurlyeq v$.

ii) $u \preccurlyeq v$ implies $au \preccurlyeq av$ for every $a \in \mathbb{R}$(every $a \in \mathbb{C}$ if E is a complex Banach space).

iii) $0 \leqslant a \leqslant b$ in $\mathbb{R}$ implies $au \preccurlyeq bu$ for every $u \in E$.

iv) If $u_1 \preccurlyeq v_1$, $u_2 \preccurlyeq v_2$ and $v_1 \preccurlyeq v_1 + v_2$, then $u_1 \preccurlyeq u_1 + u_2$ and $u_1 + u_2 \preccurlyeq v_1 + v_2$.

v) $u + v \preccurlyeq 2v$ implies $\|u\| \leqslant \|v\|$.

vi) $u_n \preccurlyeq v$ $(n \in \mathbb{N})$ and $\|u_n - u\| \to 0$ implies $u \preccurlyeq v$.

Alfsen and Effros [1] have considered the following two order relations of the above type which make sense for any Banach space :

— The relation $\preccurlyeq_L$, defined by

$$u \preccurlyeq_L v \text{ iff } \|v\| = \|u\| + \|v - u\|;$$

— The relation $\preccurlyeq_M$, defined by

$u \preccurlyeq_M v$ iff every closed ball containing 0 and v also contains u.

Observe that the definition of $\preccurlyeq_M$ can be reformulated as : $u \preccurlyeq_M v$ iff $\|w + u\| \leqslant \max\{\|w\|, \|w + v\|\}$ for every $w \in E$. While the verification of the fact that $\preccurlyeq_L$ satisfies the conditions in Definition 2.1 is immediate, the verification of condition iv) for $\preccurlyeq_M$ is less obvious ; see [1].

The systematic study of order relations of Alfsen-Effros type was started in 1983 by the first author [10] who made the observation that in every Banach lattice, the order relation $\preccurlyeq_o$ given by

$$u \preccurlyeq_o v \text{ iff } |v| = |u| + |v - u|$$

also satisfies the conditions i)—vi) in Definition 2.1. As the following result shows, $\preccurlyeq_o$ recalls both $\preccurlyeq_L$ and $\preccurlyeq_M$.

2.2. PROPOSITION. (See [11]). *Let E be a Banach lattice. Then the following assertions are equivalent for every $u, v \in E$:*

i) $u \leqslant_o v$.

ii) *Every order interval of E containing* 0 *and v also contains u.*

iii) $|w+u| \leqslant |w| \vee |w+v|$ *for every* $w \in E$.

iv) $u_- \leqslant v_-$ *and* $u_+ \leqslant v_+$.

It is perhaps worthwile to mention that the one dimensional complex Banach space $\mathbb{C}$ admits only the trivial Alfsen-Effros type order relation, namely

$$u \ll v \text{ iff } u = av \text{ for some } a \in [0, 1].$$

The relation $\ll_L$ reduces to the trivial relation precisely on strictly convex spaces.

In the remainder of this section, $\ll$ will be an order relation of Alfsen-Effros type on a Banach space E.

2.3. *Definition.* The *centralizer* associated with $\ll$ is the set $Z_{\ll}(E)$ of all linear operators U on E for which there exist $a, b \in \mathbb{R}_+$ (depending on U) such that $U(u) + au \ll bu$ for every $u \in E$.

2.4 *Definition.* A projection P on E is said to be a $\ll$ — *Cunningham projection* (or, simply, $\ll$ — projection) provided that $Pu \ll u$ for every $u \in E$.

2.5 *Definition.* A subspace of E is called a $\ll$ — *summand* provided that it is the image of a $\ll$ — Cunningham projection.

The predecessors of the concepts introduced by Definitions 2.3—2.5 are the concepts corresponding to the situations $\ll = \ll_L$ and $\ll = \ll_M$, first considered by Cunningham and Alfsen and Effros. See [1] for a complete story. The study of the above concepts in the abstract setting of an arbitrary Alfsen-Effros type order relation was initiated by the first author in [10].

We recollect here, without proofs, some facts connected with centralizers. See [12], [13] for details.

Recall that an *f-algebra* is a vector lattice A endowed with a structure of algebra such that $A_+ \cdot A_+ \subset A_+$ and the relation $a \wedge b = 0$ implies $ac \wedge b = ca \wedge b = 0$ for any $c \in A_+$. Any Archimedean f-algebra is associative and commutative.

The set $Z_{\ll}(E)$ is an algebra of bounded linear operators. The subset $Z_{\ll}(E)_+$ of $Z_{\ll}(E)$ formed by those U for which the constant a in Definition 2.3 is equal to 0 is a cone in $Z_{\ll}(E)$ such that the order relation defined by it endows $Z_{\ll}(E)$ with a structure of Archimedean f-algebra. The map 1_E is a strong order unit for $Z_{\ll}(E)$ and the norm associated to this strong order unit coincides with the operator norm on $Z_{\ll}(E)$. Consequently, $Z_{\ll}(E)$ is a commutative Banach algebra.

2.6 PROPOSITION. (See [13]). *$Z_{\ll}(E)$ is an order complete f-algebra provided that there exists a linear topology τ on E such that every $\ll$ —decreasing net has a greatest lower bound and τ — converges to it.*

In the case when E is a Banach lattice and $\ll = \ll_o$, $Z_{\ll}(E)$ coincides with the usual lattice-theoretic centralizer of E, i.e., the set of those linear operators U on E satisfying $|U(u)| \leqslant a|u|$ for all $u \in E$, with $a \in \mathbb{R}_+$ depending only on U.

The $\preccurlyeq$ — Cunningham projections are precisely the idempotents in $Z_{\preccurlyeq}(E)$; in particular, all of them commute. The set $\mathbb{P}_{\preccurlyeq}(E)$ of all such projections constitutes a Boolean algebra of projections if we put

$$P \vee Q = P + Q - PQ,$$
$$P \wedge Q = PQ,$$
$$P^{\perp} = 1_E - P.$$

Given a $\preccurlyeq$ — summand F of E, the *complementary subspace* $F^{\perp}$ of F is defined as the set

$$\{u \mid u \in E, [0, u] \cap F = \{0\}\};$$

here and elsewhere, $[u, v]$ denotes as usually the $\preccurlyeq$ — order interval $\{w \mid w \in E,\ u \preccurlyeq w \preccurlyeq v\}$. A useful remark is that if P is a projection, then

$$(\operatorname{Im} P)^{\perp} = \operatorname{Im} P^{\perp} = \operatorname{Ker} P.$$

For E a von Neumann algebra and $\preccurlyeq = \preccurlyeq_M$, the Cunningham projections are the central projections on the w'-closed two sided algebraical ideals. See [1].

For E a Banach lattice and $\preccurlyeq = \preccurlyeq_o$, the Cunningham projections are the band projections. See [11].

3. VECTOR NORMS

3.1. *Definition.* Let E be a vector space. A *vector norm* on E is a map φ defined on E with values in a vector lattice X, satisfying the following requirements:

i) $\varphi(u) \geqslant 0$ for every $u \in E$; $\varphi(u) = o$ iff $u = 0$.

ii) $\varphi(au) = |a| \cdot \varphi(u)$ for every $a \in \mathbb{R}$, $u \in E$ (every $a \in \mathbb{C}$ if E is a complex vector space).

iii) $\varphi(u + v) \leqslant \varphi(u) + \varphi(v)$ for every $u, v \in E$.

If E is a Banach space, X is a Banach lattice and φ satisfies in addition

iv) $\|\varphi(u)\| = \|u\|$ for every $u \in E$,

then φ is called an *isometric vector norm*.

3.2. *Definition.* (L. V. Kantorovich). A vector norm $\varphi : E \to X$ is said to have the *Riesz decomposition property* provided that for every $u \in E$ and every $x_1, x_2 \in X_+$ with $\varphi(u) \leqslant x_1 + x_2$, there are $u_1, u_2 \in E$ such that $u = u_1 + u_2$ and $\varphi(u_1) \leqslant x_1$, $\varphi(u_2) \leqslant x_2$.

For latter purposes we record here the following lemma:

3.3 Lemma. *Let E be a vector space, X an order complete vector lattice, $U : E \to X$ a linear map and $P_1, P_2 : E \to X$ sublinear maps such that $U(u) \leqslant P_1(u) + P_2(u)$ for every $u \in E$. Then there are linear maps $U_1, U_2 : E \to X$ such that $U = U_1 + U_2$ and $U_i(u) \leqslant P_i(u)$ for every $u \in E$, $i \in \{1, 2\}$.*

Proof. Consider the sublinear map $P : E \times E \to X$ given by $P(u_1, u_2) = P_1(u_1) + P_2(u_2)$. Let $D = \{(u, u) \mid u \in E\}$. The map $V : D \to X$ given by $V(u, u) = U(u)$ satisfies $V(u, u) \leq P(u, u)$ for every $(u, u) \in D$. Consequently, the operatorial version of the Hahn-Banach theorem (see [5] page 248) allows us to extend V to a linear map, denoted again by V, defined on the whole $E \times E$ and satisfying $V(u_1, u_2) \leq P(u_1, u_2)$ for every $(u_1, u_2) \in E \times E$. The maps U_1, U_2 defined by $U_1(u) = V(u, 0)$, $U_2(u) = V(0, u)$ have all the required properties. ∎

We indicate now some representative examples of vector norms having the Riesz decomposition property (abbreviated, RDP).

3.4 *Example.* The $\mathbb{R}$ — valued vector norms are precisely the usual norms. Clearly, any such a norm has the RDP.

3.5 *Example.* If E is a vector lattice, the map $\varphi : E \to E$ given by $\varphi(u) = |u|$ is a vector norm. The classical Riesz decomposition property for vector lattices means precisely that φ has the RDP.

3.6 *Example.* Let E be a Banach space and let X be an order complete vector lattice. A linear operator $U : E \to X$ is called ***majorizing*** if $U(B_E)$ is order bounded in X. The set of all majorizing operators from E to X is a vector space, denoted by $M(E, X)$. The map $\mu : M(E, X) \to X$ given by $\mu(U) = \sup U(B_E)$ is a vector norm having the RDP. To see this, let $\mu(U) \leq x_1 + x_2$. Lemma 3.3 applied to the linear map U and to the sublinear maps P_1, $P_2 : E \to X$ given by $P_i(u) = \|u\| \cdot x_i$ ($i \in \{1, 2\}$) yields the linear maps U_1, U_2 such that $U = U_1 + U_2$ and $U_i(u) \leq \|u\| \cdot x_i$ ($u \in E$, $i \in \{1, 2\}$). It follows that $U_i \in M(E, X)$ and $\mu(U_i) \leq x_i$.

Suppose now that X is a Banach lattice and define the norm $\| \ \|_M$ on $M(E, X)$ by $\|U\|_M = \|\mu(U)\|$. Endowed with this norm, $M(E, X)$ becomes a Banach space and μ becomes an isometric vector norm.

3.7. *Example.* Let E be a Banach space and let X be a vector lattice. A linear operator $U : X \to E$ is called *cone summable* if for every $x \in X_+$ we have

$$\text{(1)} \qquad \sup \left\{ \sum_{i=1}^{n} \| U(x_i) \| \mid n \geq 1,\ x_i \in X_+,\ \sum_{i=1}^{n} x_i = x \right\} < \infty.$$

The set of all cone summable operators from X to E is a vector space, denoted by $S_+(X, E)$. For every $U \in S_+(X, E)$, let $\sigma(U)x$ be the supremum of the set in (1). The map $x \to \sigma(U)x$ ($x \in X_+$) can be extended by linearity to a positive linear form on X, denoted by $\sigma(U)$. We have thus obtained a vector norm $\sigma : S_+(X, E) \to X^\sim$, where $X^\sim$ denotes the vector lattice of all order bounded linear forms on X.

If E is a dual Banach space, then σ has the RDP. This can be seen as follows : let F be a predual for E, i.e., $E = F'$. To every $U \in S_+(X, E)$ we associate $\overline{U} \in M(F, X^\sim)$ by $\overline{U}(v)(x) = U(x)(v)$. The correspondence $U \to \overline{U}$ establishes a bijection between $S_+(X, E)$ and $M(F, X^\sim)$ such that $\mu(\overline{U}) = \sigma(U)$; it remains to use Example 3.6 in order to conclude the proof.

Suppose now that X is a Banach lattice and define the norm $\| \ \|_S$ on $S_+(X, E)$ by $\|U\|_S = \|\sigma(U)\|$. Endowed with this norm, $S_+(X, E)$ becomes a Banach space and σ becomes an isometric vector norm.

3.8. *Example.* Let E be a Banach space and let X be a Banach lattice; we underline that X need not be order complete. Let $M_*(E', X)$

be the space of all linear operators $U : E' \to X$ satisfying the following requirements :

i) $U'(X') \subset \mathcal{I}_E(E)$.

ii) There is an $x \in X_+$ such that $U(B_{E'})$ is contained in X_x and is totally bounded for $\| \ \|_x$.

It is well known that the supremum of a totally bounded set in a Banach lattice with strong order unit always exists. Consequently, for every $U \in M_*(E', X)$ it makes sense to consider the element $\mu(U) = \sup U(B_{E'})$ of X. We have thus defined a vector norm $\mu : M_*(E', X) \to X$. With respect to the norm $\| \ \|_M$ given by $\| U \|_M = \| \mu(U) \|$, $M_*(E', X)$ becomes a Banach space and μ becomes an isometric vector norm.

The vector norm μ has the RDP. Indeed, as every order ideal X_x is order isomorphic and isometric to a space $C(K)$, it suffices to prove our assertion only in the case $X = C(K)$. But in this situation, for every $U \in M_*(E', C(K))$ there is a continuous map $F : K \to E$ such that $U(u)(t) = u(F(t))$ for every $u \in E'$ and $t \in K$. The hypothesis $\mu(U) \leqslant x_1 + x_2$ means that $\|F(t)\| \leqslant x_1(t) + x_2(t)$ for every $t \in K$. Define the continuous maps $F_i : K \to E$ by

$$F_i(t) = (x_1(t) + x_2(t))^{-1} \, x_i(t) F(t), \text{ if } x_1(t) + x_2(t) > 0$$

$$F_i(t) = 0 \qquad , \text{ if } x_1(t) + x_2(t) = 0$$

or $i \in \{1, 2\}$. The operators $U_i \in M_*(E', C(K))$ given by $U_i(u)(t) = u(F_i(t))$ $(i \in \{1 ;2\})$ satisfy al the requirements in the definition of the RDP.

The importance of the Banach space $M_*(E', X)$ lies in the fact that it is isometrically isomorphic to the M-tensor product $E \hat{\otimes}_M X$. See [4]

The interest in vector norms with the RDP is justified by the possibility of dualizing such norms. Indeed, given the Banach space E, the Banach lattice X and the isometric vector norm $\varphi : E \to X$ with the RDP, a dual vector norm $\varphi' : E' \to X'$ can be defined by

$$\varphi'(u')(x) = \sup_{\varphi(u) \leqslant x} |u'(u)|$$

for every $u' \in E'$ and $x \in X_+$; the map $\varphi'(u') : X_+ \to \mathbb{R}_+$ is positively homogeneous and additive (because of the RDP) and thus extends uniquely to a positive linear form on X, also denoted by $\varphi'(u')$. It is clear that φ' is a vector norm: in fact, we have

3.9. Proposition. *φ' is an isometric vector norm with the* RDP.

Proof. The fact that φ' is isometric is straightforward calculation. Indeed,

$$\| \varphi'(u') \| = \sup_{\substack{\|x\| \leqslant 1 \\ x \geqslant 0}} \varphi'(u')(x) = \sup_{\substack{\|x\| \leqslant 1 \\ x \geqslant 0}} \sup_{\varphi(u) \leqslant x} |u'(u)| = \sup_{\|u\| \leqslant 1} |u'(u)| = \|u'\|.$$

To see that φ' has the RDP, let $\varphi(u') \leqslant x_1' + x_2'$. Equivalently,

$$u'(u) \leqslant x_1'(\varphi(u)) + x_2'(\varphi(u)), \ u \in E.$$

By applying Lemma 3.3 to the linear form u' and to the sublinear forms $u \to x_i'(\varphi(u))$ $(i \in \{1, 2\})$, we obtain the linear forms u_i such that $u_i(u) \leqslant$

$\leqslant x_i'(\varphi(u))(u \in E, i \in \{1, 2\})$ and $u = u_1 + u_2$. Hence $\varphi'(u_i) \leqslant x_i'$ and the proof is complete. ■

Proposition 3.9 allows us in particular to consider φ'', φ''' and so on. We have the following canonical relation between φ and φ'' :

3.10. PROPOSITION. *Let* $\varphi : E \to X$ *be an isometric vector norm having the* RDP. *Then*

$$x'(\varphi(u)) = \sup_{\varphi'(u) \leqslant x'} |u'(u)|$$

for every $u \in E$ *and* $x' \in X'_+$. *In other words,* $\varphi''(\mathscr{I}_E(u)) = \mathscr{I}_X(\varphi(u))$.

Proof. The map $u \to x'(\varphi(u))$ is a seminorm on E and the set of linear forms majorated by it is precisely $\{u' | u' \in E', \ \varphi'(u') \leqslant x'\}$; thus, our assertion is a consequence of the Hahn-Banach theorem. ■

Examples of duality

The duals of the $\mathbb{R}$-valued vector norms are the usual dual norms.

The dual of the vector norm $u \to |u|$ on a Banach lattice E is the vector norm $u' \to |u'|$ on E'.

The dual of $M_*(E', F)$ can be isometrically identified with $S_+(F, E')$. When this identification is performed, one can show, by using the techniques in [4], that the dual norm of the vector norm μ on $M_*(E', F)$ is the vector norm σ on $S_+(F, E')$.

4. THE RELATIONS $\leqslant_{L,\varphi}$ AND $\leqslant_{M,\varphi}$ AND THEIR DUALITY

Throughout this section, E will denote a Banach space, X a Banach lattice and $\varphi : E \to X$ an isometric vector norm. Whenever φ' will be considered, it will be understood that φ has the RDP.

4.1. *Definition.* The relation $\leqslant_{L,\varphi}$ on E is defined by

$$u \leqslant_{L,\varphi} v \text{ iff } \quad \varphi(v) = \varphi(u) + \varphi(v - u).$$

The relation $\leqslant_{M,\varphi}$ on E is defined by

$$u \leqslant_{M,\varphi} v \text{ iff } \varphi(w + u) \leqslant \varphi(w) \vee \varphi(w + v)$$

for every $w \in E$.

4.2. PROPOSITION. $\leqslant_{L,\varphi}$ *and* $\leqslant_{M,\varphi}$ *are order relations of Alfsen-Effros type.*

Proof. The verification of conditions i)—iv) in Definition 2.1 for $\leqslant_{L,\varphi}$ is quite elementary. For instance, the verification of iv) needs only the triangle inequality. Indeed, we have by hypothesis

$$\varphi(v_i) = \varphi(u_i) + \varphi(v_i - u_i), \quad i \in \{1, 2\}$$

and

$$\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2).$$

Then

$$\varphi(v_1 + v_2) \leqslant \varphi(v_1 + v_2 - u_1 - u_2) + \varphi(u_1 + u_2) \leqslant$$
$$\leqslant \varphi(v_1 - u_1) + \varphi(u_1) + \varphi(v_2 - u_2) + \varphi(u_2) =$$
$$= \varphi(v_1) + \varphi(v_2) = \varphi(v_1 + v_2),$$

which implies that $u_1 + u_2 \ll_{L,\varphi} v_1 + v_2$ and $u_1 \ll_{L,\varphi} u_1 + u_2$.

The fact that $\ll_{M,\varphi}$ satisfies all conditions in Definition 2.1 except for iv) is also straightforward. The verification of iv) is reduced to the scalar case and proceeds as follows. We have by hypothesis $u_i \ll_{M,\varphi} v_i$ ($i \in \{1, 2\}$) and $v_1 \ll_{M,\varphi} v_1 + v_2$. Given $w \in E$, we have to prove that $\varphi(w+u_1) \leqslant \varphi(w) \vee \varphi(w + u_1 + u_2)$ and $\varphi(w + u_1 + u_2) \leqslant \varphi(w) \vee \varphi(w + v_1 + v_2)$. Let $x = \varphi(w) + \varphi(u_1) + \varphi(u_2) + \varphi(v_1) + \varphi(v_2)$. As $\varphi(u_i)$, $\varphi(v_i)$ and $\varphi(w)$ all belong to X_x ($i \in \{1, 2\}$) and X is order isomorphic to a space $C(K)$, all we have to prove is that the relations

$$\delta(\varphi(w + u_1)) \leqslant \max\{\delta(\varphi(w)), \delta(\varphi(w + u_1 + u_2))\} \tag{2}$$

$$\delta(\varphi(w + u_1 + u_2)) \leqslant \max\{\delta(\varphi(w)), \delta(\varphi(w + v_1 + v_2))\} \tag{3}$$

hold for any lattice homomorphism $\delta : X_x \to \mathbb{R}$. To this purpose, let δ be any such lattice homorphism and consider the seminorm p on $F = \varphi^{-1}(X_x)$ given by $p(u) = \delta(\varphi(u))$. The hypothesis yields that $T(u_i) \ll_M T(v_i)$ ($i \in \{1,2\}$) and $Tv_1 \ll_M Tv_1 + Tv_2$ in (F, p), where $T : F \to (F, p)$ denotes the canonical map. Consequently, we also have

$$T(u_1) \ll_M T(u_1) + T(u_2) \tag{4}$$

and

$$T(u_1) + T(u_2) \ll_M T(v_1) + T(v_2). \tag{5}$$

Now recall that the definition of $\ll_M$ makes use of an element which runs over the whole space; by taking $T(w)$ as that element in (4) and (5), one obtains precisely (2) and (3). ■

For $\varphi(u) = \|u\|$, the norm of E, we have $\ll_{L,\varphi} = \ll_L$ and $\ll_{M,\varphi} = \ll_M$.

For E a Banach lattice and $\varphi(u) = |u|$, the modulus of E, we have $\ll_{L,\varphi} = \ll_{M,\varphi} = \ll_o$; see Proposition 2.2.

We leave as an open problem the study of the relations $\ll_{L,\varphi}$ and $\ll_{M,\varphi}$ and of the concepts associated with them in the situation when φ is one of the vector norms defined in Examples 3.6—3.8.

The centralizer associated with $\ll_{L,\varphi}$ (respectively $\ll_{M,\varphi}$) will be denoted by $Z_{L,\varphi}(E)$ (respectively $Z_{M,\varphi}(E)$).

4.3. Proposition. *Let U be a bounded linear operator on E. Then $U \in Z_{L,\varphi}(E)$ (respectively $Z_{M,\varphi}(E)$) if and only if $U' \in Z_{M,\varphi'}(E')$ (respectively $Z_{L,\varphi'}(E')$).*

Proof. Suppose first that $U \in Z_{L,\varphi}(E)$. Without loosing generality, we may assume that $0 \leqslant U \leqslant 1_E$ in $Z_{L,\varphi}(E)$. Let w', $u' \in E'$ and $u \in E$ be given. By hypothesis we have $\varphi(u) = \varphi(Uu) + \varphi(u - Uu)$. The inequality

$$(w' + U'u')(u) = w'(u - Uu) + (w' + u')(Uu) \leqslant$$
$$\leqslant \varphi'(w')(\varphi(u - Uu)) + \varphi'(w' + u')(\varphi(Uu)) \leqslant$$
$$\leqslant (\varphi'(w') \vee \varphi'(w' + u'))(\varphi(u))$$

shows, by taking suprema, that

$$\varphi'(w' + U'u') \leqslant \varphi'(w') \vee \varphi'(w' + u'),$$

which means that $U'(u') \ll_{M,\varphi'} u'$, as w' was arbitrary. Hence $U' \in Z_{M,\varphi'}(E')$.

Now suppose that $U \in Z_{M,\varphi}(E)$; as above, we may assume that $0 \leqslant U \leqslant 1_E$. Let $u' \in E'$ and $v, w \in E$ be given. We have

$$(U'u')(v) + (u' - U'u')(w) = u'(Uv + w - Uw) \leqslant$$
$$\leqslant \varphi'(u')(\varphi(Uv + w - Uw)) \leqslant \varphi'(u')(\varphi(v) \vee \varphi(w))$$

as

$$\varphi(Uv + w - Uw) = \varphi(w + U(v - w)) \leqslant \varphi(w) \vee \varphi(v)$$

by hypothesis. By taking the appropriate suprema we obtain $\varphi'(U'u') + \varphi'(u' - U'u') \leqslant \varphi'(u')$. As the reverse inequality $\varphi'(u') \leqslant \varphi'(U'u') + \varphi'(u' - U'u')$ is always true, we obtain that $U'u' \ll_{L,\varphi'} u'$, hence $U' \in Z_{L,\varphi'}(E')$.

Finally, if $U' \in Z_{M,\varphi'}(E')$ (respectively $Z_{L,\varphi'}(E')$), then $U'' \in Z_{L,\varphi''}(E'')$ (respectively $Z_{M,\varphi''}(E'')$) by what have just been proved. Taking into account Proposition 3.10 we conclude that $U \in Z_{L,\varphi}(E)$ (respectively $Z_{M,\varphi}(E)$). ■

4.4. COROLLARY. *The map $U \to U'$ is an isometric, algebraic and lattice homomorphism, of $Z_{L,\varphi}(E)$ (respectively $Z_{M,\varphi}(E)$) into $Z_{M,\varphi'}(E')$ (respectively $Z_{L,\varphi'}(E')$).*

Proof. The fact that the map under consideration is well defined follows from Proposition 4.3. The lattice homomorphism part follows by combining the next two remarks. First, it was shown during the proof of Proposition 4.3 that the map $U \to U'$ takes $Z_{L,\varphi}(E)_+$ (respectively $Z_{M,\varphi}(E)_+$) into $Z_{M,\varphi'}(E')_+$ (respectively $Z_{L,\varphi'}(E')_+$). Second, in every Archimedean f-algebra with unit, the relation $a \wedge b = 0$ is equivalent to: $a \geqslant 0$, $b \geqslant 0$ and $ab = 0$. ■

4.5. PROPOSITION. *For every projection P on E the following are equivalent:*

i) *P is an $\ll_{M,\varphi}$ — projection.*

ii) *$\varphi(v + Pu) \leqslant \varphi(v) \vee \varphi(v + u)$ for every $u, v \in E$.*

iii) *$\varphi(u) = \varphi(Pu) \vee \varphi(u - Pu)$ for every $u \in E$.*

Proof. Clearly, i)$\Leftrightarrow$ii) and iii)$\Rightarrow$ii).

ii)$\Rightarrow$iii). Notice first that $\varphi(Pu), \varphi(u - Pu) \leqslant \varphi(u)$ for every $u \in E$.

On the other side,

$$\varphi(Pu + (I - P)v) = \varphi(v + P(u - v)) \leqslant \varphi(v) \vee \varphi(u)$$

for every $u, v \in E$, which implies

$$\varphi(u) = \varphi(P^2u + (I - P)^2u) \leqslant \varphi(Pu) \vee \varphi(u - Pu)$$

for every $u \in E$. ∎

4.6. PROPOSITION. *$Z_{L,\varphi'}(E')$ and $Z_{M,\varphi'}(E')$ are order complete f-algebras.*

Proof. Let $\leqslant$ be one of the relations $\leqslant_{L,\varphi'}$ and $\leqslant_{M,\varphi'}$. By Proposition 2.6, it suffices to show that every $\leqslant$ — decreasing net has a greatest lower bound and w' — converges to it. The latter fact can be obtained via standard arguments if we prove that every $\leqslant$ — order interval $[u', v']$ is w' — closed (and hence, w' — compact). By condition iv) in Definition 2.1, $[u', v'] = u' + [0, v' - u']$; in order to conclude the proof it remains to remark that the lower semicontinuity of each map $z' \to \varphi'(z')(x)$ $(x \in X_+)$ implies that $[0, v' - u']$ is w' — closed. ∎

We shall introduce now the notion of an ideal.

4.7. *Definition.* An $\leqslant_{L,\varphi}$ — *ideal* (respectively an $\leqslant_{M,\varphi}$ — *ideal*) of E is a closed subspace I of E with the property that the polar I° of I is an $\leqslant_{M,\varphi'}$ — summand (respectively an $\leqslant_{L,\varphi'}$ — summand) in E'.

The terminology is motivated by the case when $\leqslant = \leqslant_M$ first treated by Alfsen and Effros [1]. They noticed that the $\leqslant_M$ — ideals of a C^* — algebra are precisely the norm closed two sided algebraical ideals.

The case $\leqslant = \leqslant_0$ is considered below.

4.8. PROPOSITION. *Let E be a Banach lattice and let I be a closed subspace of E. Then the following assertion are equivalent:*

i) *I is an $\leqslant_0$ — ideal.*

ii) *I is an order ideal, i.e., $|y| \leqslant |x|$ and $x \in I$ imply $y \in I$.*

Returning to the abstract setting described at the beginning of this section, let us establish the conditions that must be satisfied by an ideal in order to be a summand.

4.9. PROPOSITION. *Let I be an $\leqslant_{L,\varphi}$ — ideal (respectively an $\leqslant_{M,\varphi}$ — ideal) in E. Then the following assertions are equivalent:*

i) *I is an $\leqslant_{L,\varphi}$ — summand (respectively an $\leqslant_{M,\varphi}$ — summand).*

ii) *The Cunningham projection associated with I° is w' — continuous.*

iii) *$I^{\circ\perp}$ is w' — closed.*

iv) *For every $u \in E$ there is a $v \in I$ such that $u'(u) = u'(v)$ for every $u' \in I^{\circ\perp}$.*

Proof. First of all, denote by Q the Cunningham projection onto I° and remark that $I^{\circ\perp} = \operatorname{Ker} Q$.

i) $\Rightarrow$ iv). Denoting by P the Cunningham projection onto I, we have $Q = (1_E - P)'$ by Proposition 4.3. Consequently, for every $u' \in I^{\circ\perp}$ and $u \in E$ we have

$$u'(u) = u'(Pu) + u'(u - Pu) = u'(v)$$

with $v = Pu \in I$.

iv) $\Rightarrow$ iii). Let $u'_\delta \in I^{\circ\perp}$, $u'_\delta \xrightarrow{w'} u'$. To prove that $u' \in I^{\circ\perp}$, take any $u \in E$. By hypothesis, there is a $v \in I$ such that $z'(u) = z'(v)$ for every $z' \in I^{\circ\perp}$. As u'_δ and $u' - Qu'$ belong to $I^{\circ\perp}$, it follows that

$$u'(u) = \lim_\delta u'_\delta(u) = \lim_\delta u'_\delta(v) = u'(v) =$$

$$= (Qu')(v) + (u' - Qu')(v) = (u' - Qu')(u).$$

Hence, $(Qu')(u) = 0$; as u was arbitrary, $u' \in I^{\circ\perp}$.

iii)$\Rightarrow$ii). Follows from the well known fact that a bounded projection P on E' is w'—continuous iff Im P and Ker P are w'—closed.

ii)$\Rightarrow$i) Follows from Proposition 4.3. ∎

5. A SITUATION WHEN ALL $\leqslant_{M,\varphi'}$— SUMMANDS ARE w'-CLOSED

A result due to Cunningham, Effros and Roy [7] asserts that every $\leqslant_M$-summand in a dual Banach space E' is w'-closed; consequently, every $\leqslant_L$—ideal in E is an $\leqslant_L$— summand.

On the other side, every (projection) band in the dual of a Banach lattice with order continuous norm is w'-closed; this fact was first noticed by Luxemburg and Zaanen. See [14].

It is the purpose of this section to bring together both of the above mentioned results, by deriving them as corollaries from the more general theorem stated below.

Throughout the section, E will be a Banach space, X a Banach lattice *with order continuous norm* and $\varphi : E \to X$ an isometric vector norm with the RDP.

5.1 THEOREM. *Every $\leqslant_{M,\varphi'}$—summand in E' is w'—closed.*

Proof. Let I be an $\leqslant_{M,\varphi'}$—summand in E' and let P be the Cunningham projection onto I. In order to prove that I is w'—closed it suffices to show that $u' \in I$ whenever $u'_\delta \in I$, $u'_\delta \xrightarrow{w'} u'$ and $\sup \|u'_\delta\| < \infty$; see [9]. But $u'_\delta - Pu' \xrightarrow{w'} u' - Pu'$, $u'_\delta - Pu' \in I$ and $u' - Pu' \in \operatorname{Ker} P$; hence we may assume from the beginning that $u' \in \operatorname{Ker} P$ and we have to prove that $u' = 0$

Denote by f the positive linear from $\varphi'(u')$ on X. In order to see that $f = 0$ it suffices to show that the carrier S_f of f, i.e., the disjoint complement of the set $\{x \mid x \in X, f(|x|) = 0\}$, is reduced to $\{0\}$. This is so because f is order continuous; see [6].

So, let $x \in S_f \cap X_+$ and $\varepsilon > 0$ be given. Put $M = \sup_\delta \|u'_\delta\|$. As the norm of X is order continuous, there is an $f_\varepsilon \in X'_+$ such that $(g - f_\varepsilon)_+(x) \leqslant \varepsilon$ whenever $g \in X'$ and $\|g\| \leqslant M$. See [8].

For every $n \in \mathbb{N}$ we have

$$\varphi'(u'_\delta + nu') = \varphi'(u'_\delta) \vee n\varphi'(u') = \varphi'(u'_\delta) \vee nf. \tag{6}$$

But

$$\varphi'(u'_\delta) = \varphi'(u'_\delta) \wedge f_\varepsilon + (\varphi'(u'_\delta) - f_\varepsilon)_+,$$

$$(7) \qquad \varphi'(u'_\delta) \vee nf \leqslant (f_\varepsilon + (\varphi'(u'_\delta) - f_\varepsilon)_+) \vee (nf + (\varphi'(u'_\delta) - f_\varepsilon)_+) =$$

$$= f_\varepsilon \vee nf + (\varphi'(u'_\delta) - f_\varepsilon)_+.$$

It follows from (6), (7) and the inequality $\| \varphi'(u'_\delta) \| \leqslant M$ that

$$(8) \qquad \varphi'(u'_\delta + nu')(x) \leqslant (f_\varepsilon \vee nf)(x) + \varepsilon.$$

The function $z' \to \varphi'(z')(x)$ being lower semicontinuous, we obtain from (8),

$$(1 + n)f(x) = \varphi'(u' + nu')(x) \leqslant (f_\varepsilon \vee nf)(x) + \varepsilon.$$

For every $n \in \mathbb{N}$ there is an $x_n \in [0, x]$ such that

$$f_\varepsilon(x_n) + nf(x - x_n) \geqslant (f_\varepsilon \vee nf)(x) - \varepsilon.$$

Consequently,

$$(1 + n)f(x) \leqslant f_\varepsilon(x_n) + nf(x - x_n) + 2\varepsilon$$

and finally

$$(9) \qquad f(x) \leqslant f_\varepsilon(x_n) - nf(x_n) + 2\varepsilon$$

for every $n \in \mathbb{N}$. We shall derive from (9) the inequality $f(x) \leqslant 2\varepsilon$; as ε was arbitrary and x was in S_f, this will conclude the proof of the fact that f, and hence u', are equal to 0.

Indeed, if $f_\varepsilon(x_n) - nf(x_n) < 0$ for some n, then $f(x) < 2\varepsilon$. Let us therefore suppose that $f_\varepsilon(x_n) - nf(x_n) \geqslant 0$ for all n. As $[0, x]$ is weakly compact as being an order interval in a Banach lattice with order continuous norm (see [6], [14], [15]), we may assume, by passing if necessary to a subsequence, that $x_n \to y \in [0, x]$ weakly. From $0 \leqslant f(x_n) \leqslant \frac{1}{n} f_\varepsilon(x)$ it follows that $f(y) = 0$; as $[0, x] \subset S_f$, the latter equality implies that $y = 0$. Hence $f_\varepsilon(x_n) \to 0$ and the conclusion follows from the relation

$$f(x) \leqslant f_\varepsilon(x_n) - nf(x_n) + 2\varepsilon \leqslant f_\varepsilon(x_n) + 2\varepsilon. \blacksquare$$

5.2. Corollary. *Every $\ll_{L,\varphi}$— ideal in E is an $\ll_{L,\varphi}$— summand.*

5.3. Corollary. *The map $U \to U'$ is an isometric, algebraic and order isomorphism of $Z_{L,\varphi}(E)$ onto $Z_{M,\varphi'}(E')$.*

Proof. Let A denote the image of $Z_{L,\varphi}(E)$ under the map $U \to U'$. As the f-algebra $Z_{M,\varphi'}(E')$ is order complete by Proposition 4.6, it is the closed linear hull of the set of its idempotents, i.e., the $\ll_{M,\varphi'}$-projections. As A is closed and contains all $\ll_{M,\varphi'}$-projections by Theorem 5.1, it follows that $A = Z_{M,\varphi'}(E')$. $\blacksquare$

The results mentioned at the beginning of this section are respectively derived as particular cases of Theorem 5.1 by taking φ to be the norm of a Banach space, respectively the modulus of a Banach lattice with order continuous norm.

6. $\ll$-SUMMANDS AND WEAK SEQUENTIAL COMPLETENESS

A classical result due to Lozanovskii [14] asserts that if a Banach lattice E has the property that $\mathscr{I}_E(E)$ is a band in E'', then E is weakly sequentially complete.

A more recent result due to Behrends [3] asserts that if a Banach space E has the property that $\mathscr{I}_E(E)$ is an $\ll_L$-summand in E'', then E is weakly sequentially complete.

It is the purpose of the present section to show that our general theory allows to bring together the above couple of results. The proof relies on two lemmas, which borrow some ideas from Behrends [3]. We note that unlike in [3], no appeal is made to the principle of local reflexivity.

Given a topological space K, a function $f: K \to \mathbb{R}$ and a point $t \in K$, we denote by $L_f(t)$ the intersection of all sets $\overline{f(V)}$ (i.e., the closure of $f(V)$), where V runs over all neighborhoods of t in K.

6.1. LEMMA *Let E be a Banach space and let $u'' \in E''$ be such that*

$$\|au'' + \mathscr{I}_E(u)\| = |a| \cdot \|u''\| + \|u\| \tag{10}$$

for every $a \in \mathbb{R}$ and $u \in E$. Denote by f the restriction of u'' to the topological space $B_{E'}$, endowed with the w'-topology. Then $L_f(u') = [-\|u''\|, \|u''\|]$ for every $u' \in B_{E'}$.

Proof. Clearly $L_f(u') \subset [-\|u''\|, \|u''\|]$. For the reverse inclusion, let $a \in [-\|u'\|, \|u''\|]$, let V be a beighborhood of u' in $B_{E'}$ and let $\varepsilon > 0$ be given. There are $\eta > 0$ and $u_1, \ldots, u_n \in E$ such that

$$\{v' \mid v' \in B_{E'}, |v'(u_i) - u'(u_i)| < \eta, 1 \leqslant i \leqslant n\} \subset V. \tag{11}$$

Define $g: \mathbb{R}u'' + \mathscr{I}_E(E) \to \mathbb{R}$ by $g(bu'' + \mathscr{I}_E(u)) = ab + u'(u)$. We have

$$|g(bu'' + \mathscr{I}_E(u))| \leqslant |a| \cdot |b| + |u'(u)| \leqslant |b| \cdot \|u''\| + \|u\| =$$
$$= \|bu'' + \mathscr{I}_E(u)\|$$

by (10), hence g has an extension of norm at most 1 to E'', again denoted by g. As $\mathscr{I}_{E'}(B_{E'})$ is w'-dense in $B_{E'''}$, there is a $v' \in B_{E'}$ such that

$$|g(u'') - u''(v')| < \varepsilon \tag{12}$$

and

$$|g(\mathscr{I}_E(u_i)) - v'(u_i)| < \eta, \ 1 \leqslant i \leqslant n. \tag{13}$$

But $g(\mathscr{I}_E(u_i)) = u'(u_i)$; hence (11) and (13) imply that $v' \in V$. As $g(u'') = a$ and $\varepsilon > 0$ was arbitrary, it follows from (12) that $a \in \overline{f(V)}$. As V was arbitrary, $a \in L_f(u')$. ■

6.2. LEMMA. *Let E be a Banach space, let $(u_n)_n \subset E$ and let $u'' \in E''$ be such that $\mathscr{I}_E(u_n) \xrightarrow{w'} u''$ and (10) holds for every $a \in \mathbb{R}$ and $u \in E$. Then $u'' = 0$.*

Proof. Let f be the restriction of u'' to the compact space (for the w'-topology) $B_{E'}$. The fact that $\mathscr{I}_E(u_n) \xrightarrow{w'} u''$ implies that f is a function of first Baire class, hence the set of points $u' \in B_{E'}$ at which f is continuous is nonvoid. At every such point u' we must have $L_f(u') = \{f(u')\}$; consequently, Lemma 6.1 gives $u'' = 0$. ■

6.3. THEOREM. *Let E be a Banach space, X a Banach lattice and $\varphi : E \to X$ an isometric vector norm with the RDP. Suppose that $\mathscr{I}_E(E)$ is an $\ll_{L,\varphi''}$-summand in E''. Then E is weakly sequentially complete.*

Proof. Let $(u_n)_n \subset E$ be a weak Cauchy sequence. There is $u'' \in E''$ such that $\mathscr{I}_E(u_n) \xrightarrow{w'} u''$. Denoting by P an $\ll_{L,\varphi''}$-projection onto $\mathscr{I}_E(E)$, we have $\mathscr{I}_E(u_n) - Pu'' \xrightarrow{w'} u'' - Pu''$, $\mathscr{I}_E(u_n) - Pu'' \in \mathscr{I}_E(E)$ and $u'' - Pu'' \in \operatorname{Ker} P$. Hence, it may be assumed from the beginning that $u'' \in \operatorname{Ker} P$; the proof will be concluded by showing that $u'' = 0$.

To this purpose, let $f \in X_+$ and let p be the seminorm on E given by $p(u) = f(\varphi(u))$. Denote by F the Banach space (E, p) and by $T : E \to F$ the canonical map, which is a bounded operator. Consequently,

$$\mathscr{I}_F(Tu_n) \xrightarrow{w'} T''u''. \tag{14}$$

A straightforward computation shows that

$$\| T''v'' \| = \varphi''(v'')(f) \tag{15}$$

for every $v'' \in E''$. The relation $u'' \in \operatorname{Ker} P$ implies

$$\varphi''(au'' + \mathscr{I}_E(u)) = |a| \cdot \varphi''(u'') + \varphi''(\mathscr{I}_E(u)) \tag{16}$$

for every $a \in \mathbb{R}$ and $u \in E$. Combining (15) with (16) we obtain

$$\| aT''u'' + \mathscr{I}_F(Tu) \| = |a| \cdot \| T''u'' \| + \| Tu \| \tag{17}$$

for every $a \in \mathbb{R}$ and $u \in E$. Taking into account (14), (17) and Lemma 6.2, it follows that $T''(u'') = 0$, that is, $\varphi''(u'')(f) = 0$ by (15). As f was arbitrary in X'_+, we infer that $u'' = 0$ and the proof is complete. ■

Received March 15, 1988

Department of Mathematics
University of Craiova
and
Department of Mathematics
INCREST
Bucharest, Romania

REFERENCES

1. E. Alfsen and E. G. Effros, *Structure in real Banach spaces*, Annals of Math. 96(1972), 98–173.
2. E. Behrends, *M-structure and the Banach-Stone theorem*, Lecture Notes in Math. vol. 736, Springer-Verlag, 1979.
3. E. Behrends, *The continuity space of a Banach space*, in Proc. of the Second International Conference on Operator Algebras, Ideals, and Their Applications in Theoretical Physics, Leipzig 1983, Teubner-Texte zur Mathematik, Leipzig, 1984.
4. J. Chaney, *Banach lattices of compact maps*, Math. Z., 129(1972), 1–19.
5. R. Cristescu, *Spaţii liniare ordonate şi operatori liniari*, Ed. Acad. Bucureşti, 1970.
6. R. Cristescu, *Structuri de ordine în spaţii liniare normate*, Ed. Ştiinţifică şi enciclopedică Bucureşti, 1983.
7. F. Cunningham, E. Effros and N. Roy, *M- structure in dual Banach spaces*, Israel J. Math., 14 (1973), 304–308.

8. P. G. Dodds and D. H. Fremlin, *Compact operators on Banach lattices*, Israel. J. Math. 34 (1979), 287–320.
9. N. Dunford and J. Schwartz, *Linear Operators*. Part I., Interscience Publ., New-York, 1958.
10. C. Niculescu, *Structuri de ordine pe spaţii de operatori*, in vol. Spaţii liniare ordonate topologice, n° 4 (1983), Univ. din Bucureşti.
11. C. Niculescu, *Cunningham projections associated to an Alfsen-Effros type order relation*, in Proc. The Tenth Conference on Operator Theory, Bucharest, August 26–September 5, 1985. Operator Theory, Advances and Applications, vol. 24, Birkhäuser, 1987.
12. C. Niculescu, *The centralizer associated to an Alfsen-Effros type order relation*. Part I. Analele Univ. Craiova, seria mat. fiz.-ch., XIV (1986), 1–9.
13. C. Niculescu, *The centralizer associated to an Alfsen-Effros type order relation*. Part II. Analele Univ. Craiova, seria mat. fiz.–ch., XV (1987).
14. C. Niculescu, *Probleme de slab compactitate în latici Banach*, in Structuri de ordine în analiza funcţională, vol. 1, Ed. Acad. Bucureşti, 1986.
15. H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, 1974.